

# Approximate functional equation and mean value formula for the derivatives of $L$ -functions attached to cusp forms

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## Abstract

Let  $f$  be a holomorphic cusp form of weight  $k$  with respect to the full modular group  $SL_2(\mathbb{Z})$ . We suppose that  $f$  is a normalized Hecke eigenform. Let  $L_f(s)$  be the  $L$ -function attached to the form  $f$ . Good gave the approximate functional equation and mean square formula of  $L_f(s)$ . In this paper, we shall generalize these formulas for the derivatives of  $L_f(s)$ .

## 1 Introduction

Let  $S_k$  be the space of cusp forms of even weight  $k \in \mathbb{Z}_{\geq 12}$  with respect to the full modular group  $SL_2(\mathbb{Z})$ . Let  $f \in S_k$  be a normalized Hecke eigenform, and  $a_f(n)$  the  $n$ -th Fourier coefficient of  $f$ . Set  $\lambda_f(n) = a_f(n)/n^{(k-1)/2}$ . The  $L$ -function attached to  $f$  is defined by

$$L_f(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_{p:\text{prime}} \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1} \quad (\operatorname{Re} s > 1), \quad (1.1)$$

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where  $\alpha_f(p)$  and  $\beta_f(p)$  satisfy  $\alpha_f(p) + \beta_f(p) = \lambda_f(p)$  and  $\alpha_f(p)\beta_f(p) = 1$ . Then it is well-known that the function  $L_f(s)$  is analytically continued to the whole  $s$ -plane by

$$(2\pi)^{-s-\frac{k-1}{2}}\Gamma(s+\frac{k-1}{2})L_f(s) = \int_0^\infty f(iy)y^{s+\frac{k-1}{2}-1}dy, \quad (1.2)$$

and has a functional equation

$$L_f(s) = \chi_f(s)L_f(1-s)$$

where  $\chi_f(s)$  is given by

$$\chi_f(s) = (-1)^{\frac{k}{2}}(2\pi)^{2s-1} \frac{\Gamma(1-s+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \quad (1.3)$$

$$= (-1)^{\frac{k}{2}}(2\pi)^{2\sigma-1}|t|^{1-2\sigma} e^{i(\frac{\pi}{2}(1-k)\text{sgn}(t)-2t\log\frac{|t|}{2\pi e})} (1 + O(|t|^{-1})) \quad (1.4)$$

where  $\text{sgn}(t)$  is defined by  $\text{sgn}(t) = 1$  for  $t \in \mathbb{R}_{>0}$  and  $\text{sgn}(t) = -1$  for  $t \in \mathbb{R}_{<0}$ , and (1.4) is obtained by Stirling's formula (see [3, (19)]).

Good [3] gave the approximate functional equation for  $L_f(s)$ :

$$L_f(\sigma + it) = \sum_{n \leq x} \frac{\lambda_f(n)}{n^s} + \chi_f(s) \sum_{n \leq y} \frac{\lambda_f(n)}{n^{1-s}} + O(|t|^{\frac{1}{2}-\sigma+\varepsilon})$$

where  $\varepsilon \in \mathbb{R}_{>0}$ ,  $s = \sigma + it$  such that  $\sigma \in [0, 1]$  and  $|t| \gg 1$ , and  $x, y \in \mathbb{R}_{>0}$  satisfying  $(2\pi)^2 xy = |t|^2$ . The feature of his proof of this equation is to introduce characteristic function and use the residue theorem. Moreover, he gave the mean square formula for  $L_f(s)$  using the above equation:

$$\int_1^T |L_f(\sigma + it)|^2 dt = \begin{cases} A_f T \log T + O(T), & \sigma = 1/2, \\ T \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2}{n^{2\sigma}} + O(T^{2(1-\sigma)}), & 1/2 < \sigma < 1, \\ T \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2}{n^{2\sigma}} + O(\log^2 T), & \sigma = 1, \end{cases} \quad (1.5)$$

where  $A_f$  is a positive constant depending on  $f$ .

Let  $\zeta(s)$  be the Riemann zeta function and  $\zeta'(s)$  be its first derivative. Since Speiser [6] proved that the Riemann Hypothesis (for short RH) is equivalent to the non-existence of zeros of  $\zeta'(s)$  in  $0 < \operatorname{Re} s < 1/2$ , zeros of  $\zeta'(s)$  have been interested by many researchers. Recently Aoki and Minamide [1] studied the density of zeros of  $\zeta^{(m)}(s)$  in the right hand side of critical line  $\operatorname{Re} s = 1/2$  by using Littlewood's method. However there is no result concerning zeros of derivatives of  $L$ -functions attached to cusp forms. The  $m$ -th derivative of  $L_f(s)$  is given by

$$L_f^{(m)}(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^m}{n^s} \quad (\operatorname{Re} s > 1).$$

Differentiating both sides of (1.2), we find

$$L_f^{(m)}(s) = \sum_{r=0}^m \binom{m}{r} (-1)^r \chi_f^{(m-r)}(s) L_f^{(r)}(1-s). \quad (1.6)$$

In this paper, we shall show the approximate functional equation and the mean value formula for  $L_f^{(m)}(s)$  for the purpose of studying the zero-density for  $L_f^{(m)}(s)$ .

Following [3], we shall introduce characteristic functions. Let  $\varphi$  be the real valued  $C^\infty$  function on  $[0, \infty)$  satisfying  $\varphi(\rho) = 1$  for  $\rho \in [0, 1/2]$  and  $\varphi(\rho) = 0$  for  $\rho \in [2, \infty)$ . Let  $\mathcal{R}$  be the set of these characteristic functions  $\varphi$ . Write  $\varphi_0(\rho) = 1 - \varphi(1/\rho)$ . It is clear to show that if  $\varphi \in \mathcal{R}$  then  $\varphi_0 \in \mathcal{R}$ . Let  $\varphi^{(j)}$  be the  $j$ -th derivative function of  $\varphi \in \mathcal{R}$ . Then  $\varphi^{(j)}$  becomes absolutely integrable function on  $[0, \infty)$ . Let  $\|\varphi^{(j)}\|_1$  be  $L_1$ -norm of  $\varphi^{(j)}$ , that is,  $\|\varphi^{(j)}\|_1 = \int_0^\infty |\varphi^{(j)}(\rho)| d\rho$ . For  $r \in \{0, \dots, m\}$ ,  $j \in \mathbb{Z}_{\geq 0}$ ,  $\rho \in \mathbb{R}_{>0}$  and  $s = \sigma + it$  such that  $|t| \gg 1$ , let  $\gamma_j^{(r)}(s, \rho)$  be

$$\gamma_j^{(r)}(s, \rho) = \frac{1}{2\pi i} \int_{\mathcal{F}} \frac{(\chi_f^{(r)}/\chi_f)(1-s-w)}{w(w+1)\cdots(w+j)} \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} (\rho e^{-i\frac{\pi}{2}\operatorname{sgn}(t)})^w dw$$

where  $\mathcal{F}$  is given by  $\mathcal{F} = \{-1/2 - \sigma + \sqrt{|t|}e^{i\pi\theta} \mid \theta \in (1/2, 3/2)\} \cup \{3/2 - \sigma + \sqrt{|t|}e^{i\pi\theta} \mid \theta \in (-1/2, 1/2)\} \cup \{u \pm \sqrt{|t|} \mid u \in [-1/2 - \sigma, 3/2 - \sigma]\}$ .

Then using (1.6) and the approximate formula for  $\chi_f^{(r)}(s)$  as  $|t| \rightarrow \infty$  where  $r \in \{0, \dots, m\}$ , we obtain the approximate functional equation for  $L_f^{(m)}(s)$  with characteristic functions:

**Theorem 1.1.** *For any  $m \in \mathbb{Z}_{\geq 0}$ ,  $l \in \mathbb{Z}_{\geq (k+1)/2}$ ,  $\varphi \in \mathcal{R}$ ,  $s = \sigma + it$  such that  $\sigma \in [0, 1]$  and  $|t| \gg 1$ , and  $y_1, y_2 \in \mathbb{R}_{>0}$  satisfying  $(2\pi)^2 y_1 y_2 = |t|^2$ , we have*

$$\begin{aligned} L_f^{(m)}(s) = & \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^m}{n^s} \varphi\left(\frac{n}{y_1}\right) + \\ & + \sum_{r=0}^m (-1)^r \binom{m}{r} \chi_f^{(m-r)}(s) \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^r}{n^{1-s}} \varphi_0\left(\frac{n}{y_2}\right) + R_{\varphi}(s), \end{aligned} \quad (1.7)$$

where  $R_{\varphi}(s)$  is given by

$$\begin{aligned} R_{\varphi}(s) = & \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^m}{n^s} \sum_{j=1}^l \varphi^{(j)}\left(\frac{n}{y_1}\right) \left(-\frac{n}{y_1}\right)^j \gamma_j^{(0)}\left(s, \frac{1}{|t|}\right) + \\ & + \chi_f(s) \sum_{r=0}^m (-1)^j \binom{m}{r} \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^r}{n^{1-s}} \times \\ & \times \sum_{j=1}^l \varphi_0^{(j)}\left(\frac{n}{y_2}\right) \left(-\frac{n}{y_2}\right)^j \gamma_j^{(m-r)}\left(1-s, \frac{1}{|t|}\right) + \\ & + O\left(y_1^{1-\sigma} (\log y_1)^m |t|^{-\frac{l}{2}} \|\varphi^{(l+1)}\|_1\right) + \\ & + O\left(y_2^{\sigma} \left(\sum_{r=0}^m (\log y_2)^r (\log |t|)^{m-r}\right) |t|^{1-2\sigma-\frac{l}{2}} \|\varphi_0^{(l+1)}\|_1\right). \end{aligned}$$

Introducing new functions  $\xi \notin \mathcal{R}$  and  $\psi_{\alpha} \in \mathcal{R}$  for making the main term of without characteristic function and the error term depending on  $\alpha \in \mathbb{R}_{\geq 0}$  of the approximate functional equation, replacing  $\varphi$  to  $\varphi_{\alpha}$  in Theorem 1.1, using Deligne's result (see [2]):  $|\lambda_f(n)| \leq d(n)$  and choosing  $\alpha$  to minimize the error term, we obtain the approximate functional equation for  $L_f^{(m)}(s)$ :

**Theorem 1.2.** *For any  $m \in \mathbb{Z}_{\geq 0}$  and  $s = \sigma + it$  such that  $\sigma \in [0, 1]$  and  $|t| \gg 1$ , we have*

$$\begin{aligned} L_f^{(m)}(s) &= \sum_{n \leq \frac{|t|}{2\pi}} \frac{\lambda_f(n)(-\log n)^m}{n^s} + \\ &+ \sum_{r=0}^m (-1)^r \binom{m}{r} \chi_f^{(m-r)}(s) \sum_{n \leq \frac{|t|}{2\pi}} \frac{\lambda_f(n)(-\log n)^r}{n^{1-s}} + O(|t|^{1/2-\sigma+\varepsilon}), \end{aligned} \quad (1.8)$$

where  $\varepsilon$  is an arbitrary positive number.

Using Rankin's result (see [5, (4.2.3), p.364]):

$$\sum_{n \leq x} |\lambda_f(n)|^2 = C_f x + O(x^{\frac{3}{5}}) \quad (1.9)$$

where  $C_f$  is a positive constant depending on  $f$ , the approximate formula of  $\chi_f^{(r)}(s)$  and the generalizations of Lemmas 6, 7 of [3] to estimate a double sum containing  $(\log n_1)^{r_1}(\log n_2)^{r_2}$  where  $r_1 + r_2 = r$ , we obtain the mean square for  $L_f^{(m)}(s)$ :

**Theorem 1.3.** *For any  $m \in \mathbb{Z}_{\geq 0}$  and large  $T \in \mathbb{R}_{>0}$ , we have*

$$\begin{aligned} &\int_0^T |L_f^{(m)}(\sigma + it)|^2 dt \\ &= \begin{cases} A_{f,m} T (\log T)^{2m+1} + O(T (\log T)^{2m}), & \sigma = 1/2, \\ T \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2 (\log n)^{2m}}{n^{2\sigma}} + O(T^{2(1-\sigma)} (\log T)^{2m}), & 1/2 < \sigma < 1, \\ T \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2 (\log n)^{2m}}{n^{2\sigma}} + O((\log T)^{2m+2}), & \sigma = 1, \end{cases} \end{aligned} \quad (1.10)$$

where  $A_{f,m}$  is given by

$$A_{f,m} = \left( \frac{1}{2m+1} + \sum_{r=0}^{2m} \frac{(-2)^{2m-r}}{r+1} \sum_{r_1+r_2=r} \binom{m}{r_1} \binom{m}{r_2} \right) C_f.$$

Theorems 1.1–1.3 is applied to the study of zero-density estimate for  $L_f^{(m)}(s)$  in [7]. In order to prove Theorems 1.1–1.3, we shall show preliminary lemmas in Section 2. Using these lemmas we shall give proof of Theorems 1.1–1.3 in Sections 3–5 respectively.

## 2 Preliminary Lemmas

To prove Theorem 1.1, we introduce a new function. For  $\varphi \in \mathcal{R}$ , let  $K_\varphi(w)$  be the function

$$K_\varphi(w) = w \int_0^\infty \varphi(\rho) \rho^{w-1} d\rho \quad (\operatorname{Re} w > 0).$$

Then the following fact is known:

**Lemma 2.1** ([3, p.335, Lemma 3]). *The function  $K_\varphi(w)$  is analytically continued for to the whole  $w$ -plane, and has the functional equation*

$$K_\varphi(w) = K_{\varphi_0}(-w). \quad (2.1)$$

Furthermore we have the integral representation

$$\frac{K_\varphi(w)}{w} = \frac{(-1)^{l+1}}{w(w+1) \cdots (w+l)} \int_0^\infty \varphi^{(l+1)}(\rho) \rho^{w+l} d\rho \quad (2.2)$$

for  $l \in \mathbb{Z}_{\geq 0}$ . Especially  $K_\varphi(0) = 1$ .

Next the following fact is useful for estimating the integrals (3.1),  $I'_1$  and  $I'_2$  in Section 1.1:

**Lemma 2.2** ([3, p.334, Lemma 2]). *Put  $s = \sigma + it$  and  $w = u + iv$ . For  $c_1, c_2 \in \mathbb{R}$  let  $D_1$  be the strip such that  $\sigma \in [c_1, c_2]$  and  $t \in \mathbb{R}$  in  $s$ -plane, and  $D_2$  a half-strip such that  $\sigma \in (-\infty, -1/2 - (k-1)/2)$  and  $t \in (-1, 1)$ . For fixed  $c_3, c_4 \in \mathbb{R}_{>0}$ , there exist  $c_5 \in \mathbb{R}_{>0}$  and  $c_6 \in \mathbb{R}_{>0}$  such that*

$$\left| \frac{\Gamma(s + w + \frac{k-1}{2})}{\Gamma(s + \frac{k-1}{2})} (e^{-i\frac{\pi}{2}\operatorname{sgn}(t)})^w \right|$$

$$\leq \begin{cases} c_5 \frac{(1 + |t + v|)^{\sigma+u-\frac{1}{2}+\frac{k-1}{2}}}{|t|^{\sigma-\frac{1}{2}+\frac{k-1}{2}}}, & s \in D_1, s + w \in D_1 \setminus D_2, |t| \geq c_3, \\ c_6 |t|^u, & s \in D_1, |w| \leq c_4 |t|^{1/2}. \end{cases} \quad (2.3)$$

The following fact is required to obtain the approximate formula for  $(\chi_f^{(r)}/\chi_f)(s)$ :

**Lemma 2.3.** *Let  $F$  and  $G$  be holomorphic function in the region  $D$  such that  $F(s) \neq 0$  and  $\log F(s) = G(s)$  for  $s \in D$ . Then for any fixed  $r \in \mathbb{Z}_{\geq 1}$ , there exist  $l_1, \dots, l_r \in \mathbb{Z}_{\geq 0}$  and  $C_{(l_1, \dots, l_r)} \in \mathbb{Z}_{\geq 0}$  such that*

$$\frac{F^{(r)}}{F}(s) = \sum_{1l_1 + \dots + rl_r = r} C_{(l_1, \dots, l_r)} (G^{(1)}(s))^{l_1} \dots (G^{(r)}(s))^{l_r} \quad (2.4)$$

for  $s \in D$ . Especially  $C_{(r, 0, \dots, 0)} = 1$ .

*Proof.* The case  $r = 1$  is true because of  $(F'/F)(s) = G'(s)$  for  $s \in D$ . If we assume (2.4) and  $C_{(r, 0, \dots, 0)} = 1$ , then we have

$$\begin{aligned} F^{(r+1)}(s) &= \sum_{1l_1 + \dots + rl_r = r} C_{(l_1, \dots, l_r)} \left( (F'G^{(1)l_1} \dots G^{(r)l_r})(s) + \right. \\ &\quad + l_1 (FG^{(1)l_1-1} G^{(2)l_2+1} \dots G^{(r)l_r})(s) + \dots + \\ &\quad + l_{r-1} (FG^{(1)l_1} \dots G^{(r-1)l_{r-1}-1} G^{(r)l_r+1})(s) + \\ &\quad \left. + l_r (FG^{(1)l_1} \dots G^{(r)l_r-1} G^{(r+1)})(s) \right) \\ &= F(s) \sum_{1l'_1 + \dots + (r+1)l'_{r+1} = r+1} C'_{(l'_1, \dots, l'_{r+1})} (G^{(1)l'_1} \dots G^{(r+1)l'_{r+1}})(s) \end{aligned}$$

and  $C'_{(r+1, 0, \dots, 0)} = 1 \cdot C_{(r, 0, \dots, 0)} = 1$ . Hence (2.4) is true for all  $r \in \mathbb{Z}_{\geq 1}$ .  $\square$

Using Lemma 2.3, we can get the approximate formula for  $(\chi_f^{(r)}/\chi_f)(s)$  as follows:

**Lemma 2.4.** *For any  $r \in \mathbb{Z}_{\geq 1}$ , the function  $(\chi_f^{(r)}/\chi_f)(s)$  is holomorphic in  $D = \mathbb{C} \setminus \{z \in \mathbb{C} \mid |\sigma| \geq k/2 - 1, |t| \leq 1/2\}$ . For any  $s \in D$  we have*

$$\frac{\chi_f^{(r)}}{\chi_f}(s) = \begin{cases} \left( -2 \log \frac{|t|}{2\pi} \right)^r + O\left( \frac{(\log |t|)^{r-1}}{|t|} \right), & |t| \gg 1, \\ O(1), & |t| \ll 1. \end{cases}$$

*Proof.* Apply Lemma 2.3 with  $F(s) = \chi_f(s)$  and  $G(s) = k \log i + (2s - 1) \log 2\pi + \log \Gamma(1 - s + \frac{k-1}{2}) - \log \Gamma(s + \frac{k-1}{2})$ . Then we have

$$\begin{aligned}
& G^{(1)}(s) \\
&= 2 \log 2\pi - \frac{\Gamma'}{\Gamma}(1 - s + \frac{k-1}{2}) - \frac{\Gamma'}{\Gamma}(s + \frac{k-1}{2}) \\
&= -\log(s + \frac{k-1}{2}) - \log(1 - s + \frac{k-1}{2}) + \frac{1}{2(s + \frac{k-1}{2})} + \frac{1}{2(1 - s + \frac{k-1}{2})} + \\
&\quad + 2 \log 2\pi + \int_0^\infty \frac{1/2 - \{u\}}{(u + s + \frac{k-1}{2})^2} du + \int_0^\infty \frac{1/2 - \{u\}}{(u + 1 - s + \frac{k-1}{2})^2} du \\
&= \begin{cases} -2 \log |t| + 2 \log 2\pi + O(|t|^{-1}), & |t| \gg 1, \\ O(1), & |t| \ll 1 \end{cases}
\end{aligned} \tag{2.5}$$

for  $s \in D$  where we used the following formula obtained by Stirling's formula (see [4, p.342, Theorem A.3.5]):

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} - \int_0^\infty \frac{1/2 - \{u\}}{(u + s)^2} du$$

and the following the approximate formula (see [3, p.335]):

$$\log s = \log |t| + i \frac{\pi}{2} \operatorname{sgn} t + O\left(\frac{1}{|t|}\right), \quad \frac{1}{s} = -\frac{i}{t} + O\left(\frac{1}{|t|^2}\right).$$

By differentiating both sides of (2.5), for any  $j \in \mathbb{Z}_{\geq 2}$  and  $s \in D$ ,  $G^{(j)}(s)$  is approximated as  $G^{(j)}(s) \ll 1/|t|^{j-1}$  when  $|t| \gg 1$  or  $G^{(j)}(s) \ll 1$  when  $|t| \ll 1$ . Since  $C_{(r,0,\dots,0)} = 1$ , it follows that the main term of  $(\chi_f^{(r)}/\chi_f)(s)$  becomes  $(G^{(1)}(s))^r$ .  $\square$

In order to prove Theorem 1.3, that is, to obtain the approximate formula of the mean square for  $L_f^{(m)}(s)$  as sharp as possible, we divide the characteristic function  $\varphi$  as a sum of  $\varphi_1$  and  $\varphi_2$ . For  $\varphi \in \mathcal{R}$ ,  $\delta, \delta_1 \in (0, 1/2)$  such that  $\delta < \delta_1 < \delta_2$  where  $\delta_2 = 2$ ,  $\varphi_1$  and  $\varphi_2$  are defined by

$$\varphi_1(\rho) = \begin{cases} 1, & \rho \in [0, \delta], \\ 0, & \rho \in [\delta_1, \infty), \end{cases} \quad \varphi_2(\rho) = \begin{cases} 0, & \rho \in [0, \delta], \\ 1, & \rho \in [\delta_1, 1/2], \\ \varphi(\rho), & \rho \in [1/2, \delta_2], \\ 0, & \rho \in [\delta_2, \infty), \end{cases} \tag{2.6}$$



satisfying  $(\varphi_1 + \varphi_2)(\rho) = 1$  for  $\rho \in [\delta, \delta_1]$ . Similarly for  $\varphi_0 \in \mathcal{R}$ ,  $\varphi_{01}$  and  $\varphi_{02}$  are defined by the above, where  $\delta_{01} = \delta_1$  and  $\delta_{02} = \delta_2 = 2$ . We shall generalize Lemma 7 of p.351 in [3]:

**Lemma 2.5.** Fix  $\alpha \in \mathbb{Z}_{\geq 0}$  and  $\beta \in \mathbb{R}_{\geq 0}$ .

(a) For  $X \in \{1, 01\}$ , we have

$$\begin{aligned} & \int_1^T \overline{\varphi_X \left( \frac{2\pi n}{t} \right)} \varphi_X \left( \frac{2\pi n}{t} \right) \frac{(\log \frac{t}{2\pi})^\alpha}{t^\beta} dt \\ &= \begin{cases} T^{1-\beta}(\log T)^\alpha / (1-\beta) + O((n^{1-\beta} \log n + T^{1-\beta})(\log T)^{\alpha-1}), & n \in [1, \delta T/2\pi), \beta \in [0, 1), \alpha \in \mathbb{Z}_{\geq 1}, \\ T^{1-\beta}/(1-\beta) + O(n^{1-\beta}), & n \in [1, \delta T/2\pi), \beta \in [0, 1), \alpha = 0, \\ O(|\log(T/n)|(\log T)^\alpha), & n \in [1, \delta T/2\pi), \beta = 1, \\ O((\log n)^\alpha / n^{\beta-1}), & n \in [1, \delta T/2\pi), \beta \in (1, \infty), \\ O(n^{1-\beta}(\log n)^\alpha), & n \in [\delta T/2\pi, \delta_1 T/2\pi), \\ 0, & n \in [\delta_1 T/2\pi, \infty), \end{cases} \end{aligned}$$

(b) For  $X \in \{1, 2\}$  and  $Y \in \{2, 02\}$ , we have

$$\begin{aligned} & \int_1^T \overline{\varphi_X \left( \frac{2\pi n}{t} \right)} \varphi_Y \left( \frac{2\pi n}{t} \right) \frac{(\log \frac{t}{2\pi})^\alpha}{t^\beta} dt \\ &= \begin{cases} O(n^{1-\beta}(\log n)^\alpha), & n \in [1, \delta_X T/2\pi), \\ 0, & n \in [\delta_X T/2\pi, \infty), \end{cases} \end{aligned}$$

(c) For  $X, Y \in \{1, 2, 01, 02\}$  and  $n_1 \neq n_2$ , we have

$$\begin{aligned} & \int_1^T \overline{\varphi_X \left( \frac{2\pi n_1}{t} \right)} \varphi_Y \left( \frac{2\pi n_2}{t} \right) \left( \frac{n_1}{n_2} \right)^{it} \frac{(\log \frac{t}{2\pi})^\alpha}{t^\beta} dt \\ &= \begin{cases} 0, & n_1 \in [\delta_X T/2\pi, \infty), \\ 0, & n_2 \in [\delta_Y T/2\pi, \infty), \\ \frac{(\log \frac{T}{2\pi})^\alpha}{iT^\beta} \overline{\varphi_X \left( \frac{2\pi n_1}{T} \right)} \varphi_Y \left( \frac{2\pi n_2}{T} \right) \frac{(n_1/n_2)^{iT}}{\log(n_1/n_2)} + \\ + O \left( \frac{(\log(\max\{n_1, n_2\}))^\alpha}{(\max\{n_1, n_2\})^{1+\beta}((\log(n_1/n_2))^2)} \right), & n_1, n_2: \text{otherwise}, \end{cases} \end{aligned}$$

(d) If there exist  $\alpha \in \mathbb{Z}_{\geq 0}$  and  $\beta \in \mathbb{R}_{\geq 0}$  such that  $M(t) = O((\log t)^\alpha/t^\beta)$ , then for  $X, Y \in \{1, 2, 01, 02\}$  we have

$$\begin{aligned} & \int_1^T \overline{\varphi_X\left(\frac{2\pi n_1}{t}\right)} \varphi_Y\left(\frac{2\pi n_2}{t}\right) \left(\frac{n_1}{n_2}\right)^{it} M(t) dt \\ &= \begin{cases} 0, & n_1 \in [\delta_X T/2\pi, \infty), \\ 0, & n_2 \in [\delta_Y T/2\pi, \infty), \\ O(T^{1-\beta}(\log T)^\alpha), & n_1, n_2: \text{otherwise}, \\ & \beta \in [0, 1), \\ O(|\log(T/\max\{n_1, n_2\})|(\log T)^\alpha), & n_1, n_2: \text{otherwise}, \\ & \beta = 1, \\ O((\log(\max\{n_1, n_2\}))^\alpha/(\max\{n_1, n_2\})^{\beta-1}), & n_1, n_2: \text{otherwise}, \\ & \beta \in \mathbb{R}_{>1}. \end{cases} \end{aligned}$$

(e) For  $X \in \{1, 01\}, Y \in \{1, 2, 01, 02\}$ , we have

$$\begin{aligned} & \int_1^T \overline{\varphi_X\left(\frac{2\pi n_1}{t}\right)} \varphi_Y\left(\frac{2\pi n_2}{t}\right) (n_1 n_2)^{it} \chi_f^{(\alpha)}(\sigma + it) dt \\ &= \begin{cases} 0, & n_1 \in [\delta_X T/2\pi, \infty), \\ 0, & n_2 \in [\delta_Y T/2\pi, \infty), \\ O(|\log(T/\max\{n_1, n_2\})|(\log T)^\alpha), & n_1, n_2: \text{otherwise}, \\ & \sigma = 1/2, \\ O((\log(\max\{n_1, n_2\}))^\alpha/(\max\{n_1, n_2\})^{2\sigma-1}), & n_1, n_2: \text{otherwise}, \\ & \sigma \in (1/2, 1]. \end{cases} \end{aligned}$$

*Proof.* First we consider the case  $n_1 \in [\delta_X T/2\pi, \infty)$  or  $n_2 \in [\delta_Y T/2\pi, \infty)$ . It is clear that  $\varphi_X(2\pi n_1/t) = 0$  or  $\varphi_Y(2\pi n_2/t) = 0$  for  $t \in [1, T]$ . Hence, (a)–(e) are true for the above  $n_1, n_2$ . Next we consider the case of  $n_1 \in [1, \delta_X T/2\pi)$  and  $n_2 \in [1, \delta_Y T/2\pi)$ . Then it is clear that  $2\pi n_1/\delta_X, 2\pi n_2/\delta_Y \in [1, T]$ . For  $t \in [1, 2\pi \max(n_1/\delta_X, n_2/\delta_Y))$ , we see that  $\varphi_X(2\pi n_1/t) = 0$  (if  $n_1/\delta_X \geq n_2/\delta_Y$ ) or  $\varphi_Y(2\pi n_2/t) = 0$  (if  $n_1/\delta_X \leq n_2/\delta_Y$ ). Hence,

$$\int_1^T \cdots dt = \int_{2\pi \max(\frac{n_1}{\delta_X}, \frac{n_2}{\delta_Y})}^T \cdots dt. \quad (2.7)$$

Later, we shall approximate the right-hand side of (2.7).

First we consider the condition of (a), that is,  $X, Y \in \{1, 01\}$  and  $n_1 = n_2 =: n$ . When  $n \in [\delta T/2\pi, \delta_1 T/2\pi)$ , we see that  $2\pi n/\delta \geq T$ . Then the right-hand side of (2.7) is estimated as

$$\leq \int_{\frac{2\pi n}{\delta_1}}^{\frac{2\pi n}{\delta}} \left| \varphi_X \left( \frac{2\pi n}{t} \right) \right|^2 \frac{(\log \frac{t}{2\pi})^\alpha}{t^\beta} dt \ll n^{1-\beta} (\log n)^\alpha. \quad (2.8)$$

When  $n \in [1, \delta T/2\pi)$ , we find that  $2\pi n/\delta \in [2\pi n/\delta_1, T]$  and  $\varphi_X(2\pi n/t) = 1$  for  $t \in [2\pi n/\delta, T]$ . Hence the right-hand side of (2.7) is

$$= \int_{\frac{2\pi n}{\delta_1}}^{\frac{2\pi n}{\delta}} \left| \varphi_X \left( \frac{2\pi n}{t} \right) \right|^2 \frac{(\log \frac{t}{2\pi})^\alpha}{t^\beta} dt + \int_{\frac{2\pi n}{\delta}}^T \frac{(\log \frac{t}{2\pi})^\alpha}{t^\beta} dt. \quad (2.9)$$

Here the first term of the right-hand side on (2.9) is estimated as

$$\ll n^{1-\beta} (\log n)^\alpha, \quad (2.10)$$

the second term of the right-hand side on (2.9) is

$$= \begin{cases} O(|\log(T/n)|(\log T)^\alpha), & \beta = 1, \\ T^{1-\beta}(\log T)^\alpha + O(T^{1-\beta}(\log T)^{\alpha-1}) + O(n^{1-\beta}(\log n)^\alpha), & \beta \in [0, 1), \\ O((\log n)^\alpha/n^{\beta-1}) + O((\log T)^\alpha/T^{\beta-1}), & \beta \in \mathbb{R}_{>1}. \end{cases} \quad (2.11)$$

where the following formula was used:

$$\begin{aligned} & \int_M^N \frac{(\log \frac{t}{2\pi})^\alpha}{t^\beta} dt \\ &= \begin{cases} (\log \frac{N}{M}) \left( (\log \frac{N}{2\pi})^\alpha + (\log \frac{N}{2\pi})^{\alpha-1} (\log \frac{M}{2\pi}) + \cdots + (\log \frac{M}{2\pi})^\alpha \right), & \beta = 1, \\ \sum_{r=0}^{\alpha} \frac{(-1)^r}{(1-\beta)^{r+1}} \frac{\alpha!}{(\alpha-r)!} \left( \frac{(\log \frac{N}{2\pi})^{\alpha-r}}{N^{\beta-1}} - \frac{(\log \frac{M}{2\pi})^{\alpha-r}}{M^{\beta-1}} \right), & \beta \neq 1. \end{cases} \end{aligned} \quad (2.12)$$

Therefore combining (2.7)–(2.11), we obtain (a).

Next we consider the condition (b), that is,  $Y \in \{2, 02\}$  and  $n_1 = n_2 = n$ . When  $n \in [1, \delta_X T/2\pi) \cap [1, \delta_Y T/2\pi)$ , that is,  $n \in [1, \delta_X T/2\pi)$ , we see that  $2\pi n/\delta \in [2\pi n/\delta_X, T]$  and  $\varphi_Y(2\pi n/t) = 0$  for  $t \in [2\pi n/\delta, T]$ . Then the right-hand side of (2.7) is

$$= \int_{\frac{2\pi n}{\delta_X}}^{\frac{2\pi n}{\delta}} \overline{\varphi_X\left(\frac{2\pi n}{t}\right)} \varphi_Y\left(\frac{2\pi n}{t}\right) \frac{(\log \frac{t}{2\pi})^\alpha}{t^\beta} dt \ll n^{1-\beta} (\log n)^\alpha. \quad (2.13)$$

From (2.7) and (2.13), (b) is obtained.

We consider the condition of (c), that is,  $n_1 \neq n_2$ ,  $n_1 \in [1, \delta_X T/2\pi)$  and  $n_2 \in [1, \delta_Y T/2\pi)$ . By integral by parts, the right-hand side of (2.7) is

$$\begin{aligned} & \overline{\varphi_X\left(\frac{2\pi n_1}{T}\right)} \varphi_Y\left(\frac{2\pi n_1}{T}\right) \frac{(\log T)^\alpha}{T^\beta} \frac{(n_1/n_2)^{iT}}{i \log(n_1/n_2)} + \\ & + \left( \overline{\varphi_X\left(\frac{2\pi n_1}{t}\right)} \varphi_Y\left(\frac{2\pi n_2}{t}\right) \frac{(\log t)^\alpha}{t^\beta} \right)'_{t=T} \frac{(n_1/n_2)^{iT}}{(\log(n_1/n_2))^2} - \\ & - \frac{1}{(\log(n_1/n_2))^2} \int_{2\pi \max(\frac{n_1}{\delta_X}, \frac{n_2}{\delta_Y})}^T \left( \overline{\varphi_X\left(\frac{2\pi n_1}{t}\right)} \varphi_Y\left(\frac{2\pi n_2}{t}\right) \frac{(\log t)^\alpha}{t^\beta} \right)'' \times \\ & \times \left( \frac{n_1}{n_2} \right)^{it} dt. \end{aligned} \quad (2.14)$$

Since  $(\varphi_X(2\pi n/t))' = O(n/t^2)$  and  $(\varphi_X(2\pi n/t))'' = O(n/t^3) + O(n^2/t^4)$  for  $X \in \{1, 2, 01, 02\}$ , it follows that

$$\begin{aligned} (\cdots)'_{t=T} & \ll (n_1 + n_2) \frac{(\log T)^\alpha}{T^{\beta+2}} + \frac{(\log T)^{\alpha-1}}{T^{\beta+1}} + \frac{(\log T)^\alpha}{T^{\beta+1}} \ll \frac{(\log T)^\alpha}{T^{\beta+1}}, \\ (\cdots)'' & \ll (n_1 + n_2) \frac{(\log t)^\alpha}{t^{\beta+3}} + (n_1^2 + n_2^2) \frac{(\log t)^\alpha}{t^{\beta+4}} + n_1 n_2 \frac{(\log t)^\alpha}{t^{\beta+4}} \ll \frac{(\log t)^\alpha}{t^{\beta+2}}. \end{aligned}$$

Hence the second term of the right-hand side of (2.14) is estimated as

$$\ll \frac{(\log T)^\alpha}{T^{\beta+1} (\log(n_1/n_2))^2} \ll \frac{(\log \max(n_1, n_2))^\alpha}{(\max(n_1, n_2))^{\beta+1} (\log(n_1/n_2))^2}, \quad (2.15)$$

and the third term of the right-hand side of (2.14) is estimated as

$$\ll \frac{1}{(\log(n_1/n_2))^2} \int_{2\pi \max(\frac{n_1}{\delta_X}, \frac{n_2}{\delta_Y})}^T \frac{(\log t)^\alpha}{t^{\beta+2}} dt \ll \frac{(\log \max(n_1, n_2))^\alpha}{(\max(n_1, n_2))^{\beta+1} (\log(n_1/n_2))^2}. \quad (2.16)$$

Combining (2.7) and (2.14)–(2.16), we obtain (c).

Next we consider the condition of (d), that is,  $n_1 \in [1, \delta_X T/2\pi)$  and  $n_2 \in [1, \delta_Y T/2\pi)$ . Then (2.12) gives that the right-hand side of (2.7) is estimated as

$$\begin{aligned} & \ll \int_{2\pi \max(\frac{n_1}{\delta_X}, \frac{n_2}{\delta_Y})}^T \frac{(\log t)^\alpha}{t^\beta} dt \\ & \ll \begin{cases} T^{1-\beta} (\log T)^\alpha, & \beta \in [0, 1), \\ |\log(T/\max(n_1, n_2))| (\log T)^\alpha, & \beta = 1, \\ (\log \max(n_1, n_2)^\alpha) / (\max(n_1, n_2))^{\beta-1}, & \beta \in \mathbb{R}_{>1}. \end{cases} \end{aligned}$$

Thus (d) is obtained.

Finally we consider the condition of (e), that is,  $X \in \{1, 01\}$ ,  $n_1 \in [1, \delta_X T/2\pi)$  and  $n_2 \in [1, \delta_Y T/2\pi)$ . Using (1.4) and Lemma 2.4, we have

$$\begin{aligned} (n_1 n_2)^{it} \chi_f^{(\alpha)}(s) &= (-1)^{-\frac{k}{2}} (-2)^\alpha (2\pi)^{2\sigma-1} e^{i\frac{\pi}{2}(1-k)\text{sgn}(t)} \times \\ & \times e^{-2t \log \frac{|t|}{2\pi e \sqrt{n_1 n_2}}} |t|^{1-2\sigma} \left( \log \frac{|t|}{2\pi} \right)^\alpha + M_1(t) \end{aligned} \quad (2.17)$$

where  $M_1(t) = O((\log |t|)^\alpha / |t|^{2\sigma})$ . Since we have  $\delta_X \delta_Y < 1$ , it follows that

$$2\pi \max(n_1/\delta_X, n_2/\delta_Y) \geq 2\pi \sqrt{(n_1 n_2)/(\delta_X \delta_Y)} > 2\pi \sqrt{n_1 n_2}.$$

Therefore we see that  $|\log(2\pi \sqrt{n_1 n_2}/t)| > -\log(\sqrt{\delta_X \delta_Y}) > 0$  and

$$e^{-i2t \log \frac{t}{2\pi e \sqrt{n_1 n_2}}} = \left( \frac{e^{-i2t \log \frac{t}{2\pi e \sqrt{n_1 n_2}}}}{2i \log \frac{2\pi \sqrt{n_1 n_2}}{t}} \right)' - \frac{e^{-i2t \log \frac{t}{2\pi e \sqrt{n_1 n_2}}}}{2it (\log \frac{2\pi \sqrt{n_1 n_2}}{t})^2} \quad (2.18)$$

for  $t \in [2\pi \max(n_1/\delta_X, n_2/\delta_Y), T]$ . By (2.17) and (2.18), the right-hand side of (2.7) is estimated as

$$\begin{aligned} &= (-1)^{-\frac{k}{2}} (-2)^\alpha (2\pi)^{2\sigma-1} e^{i\frac{\pi}{2}(1-k)} \times \\ & \times \int_{2\pi \max(\frac{n_1}{\delta_X}, \frac{n_2}{\delta_Y})}^T \overline{\varphi_X \left( \frac{2\pi n}{t} \right)} \varphi_Y \left( \frac{2\pi n}{t} \right) \frac{(\log \frac{t}{2\pi})^\alpha}{t^{2\sigma-1}} \left( \frac{e^{-i2t \log \frac{t}{2\pi e \sqrt{n_1 n_2}}}}{2i \log \frac{2\pi \sqrt{n_1 n_2}}{t}} \right)' dt \end{aligned}$$

$$+ O \left( \int_{2\pi \max(\frac{n_1}{\delta_X}, \frac{n_2}{\delta_Y})}^T \overline{\varphi_X \left( \frac{2\pi n}{t} \right)} \varphi_Y \left( \frac{2\pi n}{t} \right) M_2(t) dt \right), \quad (2.19)$$

where  $M_2(t) = O((\log t)^\alpha / t^{2\sigma})$ . From (d), the second term of the right-hand side of (2.19) is estimated as

$$\ll \begin{cases} |\log(T / \max(n_1, n_2))| (\log T)^\alpha, & \sigma = 1/2, \\ (\log \max(n_1, n_2))^\alpha / (\max(n_1, n_2))^{2\sigma-1}, & \sigma \in (1/2, 1]. \end{cases} \quad (2.20)$$

Integration by parts and (2.12) give that the first term of the right-hand side of (2.19) is

$$\begin{aligned} &= \frac{(-2)^\alpha (2\pi)^{2\sigma-1}}{(-1)^{\frac{k}{2}} e^{i\frac{\pi}{2}(k-1)}} \left( \overline{\varphi_X \left( \frac{2\pi n}{T} \right)} \varphi_Y \left( \frac{2\pi n}{T} \right) \frac{(\log \frac{T}{2\pi})^\alpha}{T^{2\sigma-1}} \frac{e^{-i2t \log \frac{T}{2\pi e \sqrt{n_1 n_2}}}}{2i \log \frac{2\pi \sqrt{n_1 n_2}}{T}} + \right. \\ &\quad \left. - \int_{2\pi \max(\frac{n_1}{\delta_X}, \frac{n_2}{\delta_Y})}^T \left( \overline{\varphi_X \left( \frac{2\pi n}{t} \right)} \varphi_Y \left( \frac{2\pi n}{t} \right) \frac{(\log \frac{t}{2\pi})^\alpha}{t^{2\sigma-1}} \right)' \frac{e^{-i2t \log \frac{t}{2\pi e \sqrt{n_1 n_2}}}}{2i \log \frac{2\pi \sqrt{n_1 n_2}}{t}} dt \right) \\ &\ll \frac{(\log T)^\alpha}{T^{2\sigma-1}} + \int_{2\pi \max(\frac{n_1}{\delta_X}, \frac{n_2}{\delta_Y})}^T \frac{(\log t)^\alpha}{t^{2\sigma}} dt \\ &\ll \begin{cases} |\log(T / \max(n_1, n_2))| (\log T)^\alpha, & \sigma = 1/2, \\ (\log \max(n_1, n_2))^\alpha / (\max(n_1, n_2))^{2\sigma-1}, & \sigma \in (1/2, 1], \end{cases} \quad (2.21) \end{aligned}$$

where the following estimate was used:

$$(\cdots)' \ll (n_1 + n_2) \frac{(\log t)^\alpha}{t^{2\sigma+1}} + \frac{(\log t)^\alpha}{t^{2\sigma}} + \frac{(\log t)^{\alpha-1}}{t^{2\sigma}} \ll \frac{(\log t)^\alpha}{t^{2\sigma}}.$$

Combining (2.7) and (2.19)–(2.21), we get (e).  $\square$

After using Lemma 2.5, we shall estimate the following sums:

**Lemma 2.6.** *For  $x \in \mathbb{R}_{\geq 2}$ ,  $r_1, r_2 \in \mathbb{Z}_{\geq 0}$  and complex valued arithmetic functions  $\alpha, \beta$  such that  $\alpha(n) \ll |\lambda_f(n)|$ ,  $\beta(n) \ll |\lambda_f(n)|$ , we have*

$$\begin{aligned} (a) \quad & \sum_{n_1 \leq n_2 \leq x} \frac{|\lambda_f(n_1) \lambda_f(n_2)| (\log n_1)^{r_1} (\log n_2)^{r_2}}{(n_1 n_2)^\sigma} \\ & \ll \begin{cases} x^{2(1-\sigma)} (\log x)^{r_1+r_2}, & \sigma \in [1/2, 1), \\ (\log x)^{r_1+r_2+2}, & \sigma = 1, \end{cases} \end{aligned}$$

$$(b) \sum_{n_1 \leq n_2 \leq x} \frac{|\lambda_f(n_1)\lambda_f(n_2)|(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1 n_2)^\sigma} \left| \log \frac{x}{n_2} \right| \ll \frac{(\log x)^{r_1+r_2}}{x^{2(\sigma-1)}} \\ \text{for } \sigma \in [1/2, 1),$$

$$(c) \sum_{n_1 < n_2 \leq x} \frac{|\alpha(n_1)\beta(n_2)|(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1 n_2)^\sigma n_2 (\log(n_1/n_2))^2} \\ \ll \begin{cases} x^{2(1-\sigma)}(\log x)^{r_1+r_2}, & \sigma \in [1/2, 1), \\ (\log x)^{r_1+r_2+2}, & \sigma = 1, \end{cases}$$

$$(d) \sum_{\substack{n_1, n_2 \leq x, \\ n_1 \neq n_2}} \frac{\overline{\alpha(n_1)}\beta(n_2)(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1 n_2)^\sigma \log(n_1/n_2)} \\ \ll \begin{cases} x^{2(1-\sigma)}(\log x)^{r_1+r_2}, & \sigma \in [1/2, 1), \\ (\log x)^{r_1+r_2+2}, & \sigma = 1. \end{cases}$$

*Proof.* Using the fact  $(\log n_1)^{r_1}(\log n_2)^{r_2} \ll (\log x)^{r_1+r_2}$  for  $n_1, n_2 \leq x$  and the estimates of  $R_\sigma(x)$  and  $S_\sigma(x)$  in [3, p.348, LEMMA 6], we obtain (a) and (b). By the same discussion for  $T_\sigma(x)$  and  $U_\sigma(x)$  with  $\alpha_{n_1} = \alpha(n_1)(\log n_1)^{r_1}$ ,  $\beta_{n_2} = \beta(n_2)(\log n_2)^{r_2}$ ,  $a_{n_1} = \lambda_f(n_1)(\log n_2)^{r_1}$ ,  $b_{n_2} = \lambda_f(n_2)(\log n_2)^{r_2}$  in [3, p.348, LEMMA 6], (c) and (d) are obtained.  $\square$

### 3 Proof of Theorem 1.1

First we shall show the following formula:

**Proposition 3.1.** *For  $s = \sigma + it$  such that  $\sigma \in [0, 1]$  and  $|t| \gg 1$ ,  $\varphi \in \mathcal{R}$ ,  $x \in \mathbb{R}_{>0}$ , and fixed  $l \in \mathbb{Z}_{\geq (l+1)/2}$ , we have*

$$L_f^{(m)}(s) = G_m(s, x; \varphi) + \chi_f(s) \sum_{r=0}^m (-1)^r \binom{m}{r} G_r \left( 1 - s, \frac{1}{x}; \varphi_0 \right),$$

where  $G_r(s, x; \varphi)$  ( $r \in \{0, \dots, m\}$ ) are given by

$$G_r(s, x; \varphi) = \frac{1}{2\pi i} \int_{(\frac{3}{2}-\sigma)} \frac{\chi_f^{(m-r)}}{\chi_f} (1-s-w) L_f^{(r)}(s+w) \frac{K_\varphi(w)}{w} \times$$

$$\times \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \left( \frac{x}{2\pi} e^{-i\frac{\pi}{2}\operatorname{sgn} t} \right)^w dw.$$

*Proof.* First we shall show that the integral

$$\frac{1}{2\pi i} \int_{-\frac{1}{2}-\sigma\pm iv}^{\frac{3}{2}-\sigma\pm iv} L_f^{(m)}(s+w) \frac{K_\varphi(w)}{w} \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \left( \frac{x}{2\pi} e^{-i\frac{\pi}{2}\operatorname{sgn} t} \right)^w dw \quad (3.1)$$

vanishes as  $|v| \rightarrow \infty$  for  $l \in \mathbb{Z}_{\geq(k+1)/2}$ . Write  $w = u + iv$  and choose  $|v| \gg |t| + 1$ , then  $|s+w| \gg |t+v| \gg 1$ . Using (1.4), (1.6) and Lemma 2.4 we have

$$L_f^{(m)}(s) \ll \sum_{r=0}^m |t|^{1-2\sigma} (\log |t|)^{m-r} |L_f^{(r)}(1-s)| \ll |t|^{1-2\sigma} (\log |t|)^m \quad (|t| \rightarrow \infty)$$

for  $\operatorname{Re} s < 0$ . Hence the Phragmén-Lindelöf theorem gives

$$\begin{aligned} L_f^{(m)}(s+w) &\ll |t+v|^{\frac{3}{2}-(\sigma+u)} (\log |t+v|)^m \\ &\ll |v|^{\frac{3}{2}-(\sigma+u)} (\log |v|)^m \quad (|v| \rightarrow \infty) \end{aligned} \quad (3.2)$$

uniformly for  $\sigma + u \in [-1/2, 3/2]$ . Using (2.2) and (2.3) we see that

$$\begin{aligned} &\frac{K_\varphi(w)}{w} \times \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \left( \frac{x}{2\pi} e^{-i\frac{\pi}{2}\operatorname{sgn} t} \right)^w \\ &\ll \frac{\|\varphi^{(l+1)}\|_1}{|v|^{l+1}} \times \frac{(1+|t+v|)^{\sigma+u-\frac{1}{2}+\frac{k-1}{2}}}{|t|^{\sigma-\frac{1}{2}+\frac{k-1}{2}}} \ll |v|^{\sigma+u-\frac{3}{2}+\frac{k-1}{2}-l} \quad (|v| \rightarrow \infty) \end{aligned} \quad (3.3)$$

uniformly for  $\sigma + u \in [-1/2, 3/2]$ . From (3.2) and (3.3), the integral (3.1) is  $\ll |v|^{\frac{k-1}{2}-l} (\log |v|)^m$ , that is, (3.1) tends to 0 as  $|v| \rightarrow \infty$  when  $l \in \mathbb{Z}_{\geq(k+1)/2}$ .

Using the above fact,  $K_\varphi(0) = 1$  and applying Cauchy's residue theorem, we have

$$\begin{aligned} L_f^{(m)}(s) &= \frac{1}{2\pi i} \left( \int_{(\frac{3}{2}-\sigma)} - \int_{(-\frac{1}{2}-\sigma)} \right) L_f^{(m)}(s+w) \frac{K_\varphi(w)}{w} \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \times \\ &\quad \times \left( \frac{x}{2\pi} e^{-i\frac{\pi}{2}\operatorname{sgn} t} \right)^w dw. \end{aligned} \quad (3.4)$$



for  $l \in \mathbb{Z}_{\geq (k+1)/2}$ . Clearly, the first term of the right-hand side of (3.4) is

$$= G_m(s, x; \varphi). \quad (3.5)$$

We consider the second term of the right-hand side of (3.4). Now we can calculate

$$\begin{aligned} & L_f^{(m)}(s+w) \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \\ &= \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \chi_f(s+w) \sum_{r=0}^m (-1)^r \binom{m}{r} \frac{\chi_f^{(m-r)}(s+w)}{\chi_f} L_f^{(r)}(1-s-w) \\ &= \chi_f(s) (2\pi)^{2w} \frac{\Gamma(1-s+w+\frac{k-1}{2})}{\Gamma(1-s+\frac{k-1}{2})} \sum_{r=0}^m (-1)^r \binom{m}{r} \frac{\chi_f^{(m-r)}(s+w)}{\chi_f} \times \\ & \quad \times L_f^{(r)}(1-s-w) \end{aligned} \quad (3.6)$$

where we used (1.3) and (1.6) which give that

$$\frac{\chi_f(s+w)}{\chi_f(s)} = (2\pi)^{2w} \frac{\Gamma(s+w)}{\Gamma(s+\frac{k-1}{2})} \frac{\Gamma(1-s-w+\frac{k-1}{2})}{\Gamma(1-s+\frac{k-1}{2})}.$$

Using (2.1), (3.6) and transforming  $w \mapsto -w$ , we see that the second term of the right-hand side of (3.4) is

$$\begin{aligned} &= -\frac{\chi_f(s)}{2\pi i} \int_{-(\frac{1}{2}+\sigma)} \frac{K_\varphi(-w)}{-w} \frac{\Gamma(1-s+w+\frac{k-1}{2})}{\Gamma(1-s+\frac{k-1}{2})} (2\pi x e^{-i\frac{\pi}{2}\text{sgn}(t)})^{-w} \times \\ & \quad \times \sum_{r=0}^m (-1)^r \binom{m}{r} \frac{\chi_f^{(m-r)}(s-w)}{\chi_f} L_f^{(r)}(1-s+w) (-dw) \\ &= \chi_f(s) \sum_{r=0}^m (-1)^r \binom{m}{r} G_r\left(1-s, \frac{1}{x}; \varphi_0\right). \end{aligned} \quad (3.7)$$

By (3.4)–(3.7) Proposition 3.1 is showed.  $\square$

Next, the approximate formula of  $G_r(s, x; \varphi)$  is written as follows:

**Proposition 3.2.** *For  $s = \sigma + it$  such that  $\sigma \in [0, 1]$  and  $|t| \gg 1$ ,  $\varphi \in \mathcal{R}$ ,  $x, y \in \mathbb{R}_{>0}$  satisfying  $x/(2\pi y) = 1/|t|$ , fixed  $r \in \{0, \dots, m\}$  and  $l \in \mathbb{Z}_{\geq (k+1)/2}$ , we have*

$$G_r(s, x; \varphi) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^r}{n^s} \sum_{j=0}^l \varphi^{(j)}\left(\frac{n}{y}\right) \left(-\frac{n}{y}\right)^j \gamma_j^{(m-r)}\left(s, \frac{1}{|t|}\right) + \\ + O(y^{1-\sigma}(\log y)^r(\log |t|)^{m-r}|t|^{-\frac{l}{2}}\|\varphi^{(l+1)}\|_1).$$

*Proof.* First using (2.2) and dividing the series  $L_f^{(r)}(s+w)$  into two path at  $\rho y$ , we can write

$$G_r(s, x; \varphi) = I_1 + I_2, \quad (3.8)$$

where  $I_1$  and  $I_2$  are given by

$$I_1 = \frac{1}{2\pi i} \int_{(\frac{3}{2}-\sigma)} \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \left(\frac{x}{2\pi} e^{-i\frac{\pi}{2}\operatorname{sgn} t}\right)^w \frac{(-1)^l}{w(w+1)\cdots(w+l)} \times \\ \times \frac{\chi_f^{(m-r)}}{\chi_f} (1-s-w) \left( \int_0^{\infty} \varphi^{(l+1)}(\rho) \rho^{w+l} \sum_{n \leq \rho y} \frac{\lambda_f(n)(-\log n)^r}{n^{s+w}} d\rho \right) dw, \\ I_2 = \frac{1}{2\pi i} \int_{(\frac{3}{2}-\sigma)} \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \left(\frac{x}{2\pi} e^{-i\frac{\pi}{2}\operatorname{sgn} t}\right)^w \frac{(-1)^l}{w(w+1)\cdots(w+l)} \times \\ \times \frac{\chi_f^{(m-r)}}{\chi_f} (1-s-w) \left( \int_0^{\infty} \varphi^{(l+1)}(\rho) \rho^{w+l} \sum_{n > \rho y} \frac{\lambda_f(n)(-\log n)^r}{n^{s+w}} d\rho \right) dw.$$

Let  $L_{\pm 1}, L_{\pm 2}, C_1, C_2$  be paths of integration defined by

$$L_{\pm 1} = \{-1/2 - \sigma \pm iv \mid v \in (\sqrt{|t|}, \infty)\}, \\ L_{\pm 2} = \{3/2 - \sigma \pm iv \mid v \in (\sqrt{|t|}, \infty)\}, \\ C_1 = \{-1/2 - \sigma + \sqrt{|t|}e^{-i\pi\theta} \mid \theta \in (1/2, 3/2)\}, \\ C_2 = \{3/2 - \sigma + \sqrt{|t|}e^{i\pi\theta} \mid \theta \in (-1/2, 1/2)\}.$$

Then by the residue theorem, we have

$$I_1 = I'_1 + \operatorname{Res} \mathcal{F}, \quad I_2 = I'_2, \quad (3.9)$$

where  $I'_1, I'_2, \text{Res } \mathcal{F}$  are given by

$$\begin{aligned}
I'_1 &= \frac{1}{2\pi i} \int_{L_{-1}+C_1+L_{+1}} \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \left( \frac{x}{2\pi} e^{-i\frac{\pi}{2}\text{sgn } t} \right)^w \times \\
&\quad \times \frac{(-1)^l}{w(w+1)\cdots(w+l)} \frac{\chi_f^{(m-r)}}{\chi_f} (1-s-w) \times \\
&\quad \times \left( \int_0^\infty \varphi^{(l+1)}(\rho) \rho^{w+l} \sum_{n \leq \rho y} \frac{\lambda_f(n)(-\log n)^r}{n^{s+w}} d\rho \right) dw, \\
I'_2 &= \frac{1}{2\pi i} \int_{L_{-2}+C_2+L_{+2}} \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \left( \frac{x}{2\pi} e^{-i\frac{\pi}{2}\text{sgn } t} \right)^w \times \\
&\quad \times \frac{(-1)^l}{w(w+1)\cdots(w+l)} \frac{\chi_f^{(m-r)}}{\chi_f} (1-s-w) \times \\
&\quad \times \left( \int_0^\infty \varphi^{(l+1)}(\rho) \rho^{w+l} \sum_{n > \rho y} \frac{\lambda_f(n)(-\log n)^r}{n^{s+w}} d\rho \right) dw, \\
\text{Res } \mathcal{F} &= \sum_{w=0, -1, \dots, -l} \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \left( \frac{x}{2\pi} e^{-i\frac{\pi}{2}\text{sgn } t} \right)^w \frac{(-1)^l}{w(w+1)\cdots(w+l)} \times \\
&\quad \times \frac{\chi_f^{(m-r)}}{\chi_f} (1-s-w) \left( \int_0^\infty \varphi^{(l+1)}(\rho) \rho^{w+l} \sum_{n \leq \rho y} \frac{\lambda_f(n)(-\log n)^r}{n^{s+w}} d\rho \right)
\end{aligned}$$

By the same way to [3, p.337, Lemma 4 (ii)],  $\text{Res } \mathcal{F}$  is written by

$$\text{Res } \mathcal{F} = \sum_{n \leq 2y} \frac{\lambda_f(n)(-\log n)^r}{n^s} \sum_{j=0}^l \varphi^{(j)} \left( \frac{n}{y} \right) \left( -\frac{n}{y} \right)^j \gamma_j^{(m-r)} \left( s, \frac{1}{|t|} \right) \quad (3.10)$$

under the condition  $x/(2\pi y) = 1/|t|$ .

Next to estimate  $I'_1$  and  $I'_2$ , we consider these integral. Clearly (2.3) gives

$$\begin{aligned}
&\frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \left( \frac{x}{2\pi} e^{-i\frac{\pi}{2}\text{sgn } t} \right)^w \\
&\ll \begin{cases} |t|^{\frac{1}{2}-\sigma-\frac{k-1}{2}} (1+|t+v|)^{\sigma+u-\frac{1}{2}+\frac{k-1}{2}} (x/2\pi)^u, & w \in L_{\pm 1, \pm 2}, \\ |t|^u (x/2\pi)^u, & w \in \mathcal{F} \end{cases} \quad (3.11)
\end{aligned}$$

as  $|t| \rightarrow \infty$ . Using Cauchy's inequality and (1.9), we have

$$\begin{aligned} \sum_{n \leq \rho y} \frac{\lambda_f(n)(-\log n)^r}{n^{s+w}} &\ll \sqrt{\sum_{n \leq \rho y} |\lambda_f(n)|^2} \sqrt{\sum_{n \leq \rho y} \frac{(\log n)^{2r}}{n^{2(\sigma+u)}}} \\ &\ll (\rho y)^{1-(\sigma+u)} (\log \rho y)^r, \quad w \in L_{\pm 1} \cup C_1, \\ \sum_{n > \rho y} \frac{\lambda_f(n)(-\log n)^r}{n^{s+w}} &\ll \int_{\rho y}^{\infty} \left( \frac{(\log \mu)^r}{\mu^{\sigma+u}} \right)' \sum_{n \leq \mu} |\lambda_f(n)| d\mu \\ &\ll (\rho y)^{1-(\sigma+u)} (\log \rho y)^r, \quad w \in L_{\pm 2} \cup C_2. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \int_0^{\infty} \varphi^{(l+1)}(\rho) \rho^{w+l} \sum_{n \leq \rho y} \frac{\lambda_f(n)(-\log n)^r}{n^{s+w}} d\rho &\ll y^{1-(\sigma+u)} (\log y)^r \|\varphi^{(l+1)}\|_1, \\ &w \in L_{\pm 1} \cup C_1, \end{aligned} \tag{3.12}$$

$$\begin{aligned} \int_0^{\infty} \varphi^{(l+1)}(\rho) \rho^{w+l} \sum_{n > \rho y} \frac{\lambda_f(n)(-\log n)^r}{n^{s+w}} d\rho &\ll y^{1-(\sigma+u)} (\log y)^r \|\varphi^{(l+1)}\|_1, \\ &w \in L_{\pm 2} \cup C_2. \end{aligned} \tag{3.13}$$

Therefore Lemma 2.4 gives

$$\frac{(-1)^l}{w \cdots (w+l)} \frac{\chi_f^{(m-r)}}{\chi_f} (1-s-w) \ll \begin{cases} |v|^{-(l+1)} (\log |v|)^{m-r}, & w \in L_{\pm 1, \pm 2}, \\ |t|^{-\frac{l+1}{2}} (\log |t|)^{m-r}, & w \in \mathcal{F}. \end{cases} \tag{3.14}$$

**Remark 3.3.** Note that

$$\begin{aligned} &\gamma_j^{(r)}(s, 1/|t|) \\ &= \begin{cases} O\left(\frac{(\log |t|)^r}{|t|^{j/2}}\right), & j \in \mathbb{Z}_{\geq 0}, \\ \frac{\chi_f^{(r)}}{\chi_f} (1-s) = \left(-2 \log \frac{|t|}{2\pi}\right)^r + O\left(\frac{(\log |t|)^{r-1}}{|t|}\right), & j = 0, \\ \frac{\chi_f^{(r)}}{\chi_f} (1-s) - \frac{\chi_f^{(r)}}{\chi_f} (-s) \frac{it}{s + \frac{k-1}{2}} = O\left(\frac{(\log |t|)^r}{|t|}\right), & j = 1, \end{cases} \end{aligned}$$

by using (3.14), the residue theorem and Lemma 2.4.

Finally combining (3.11)–(3.14) and using the same way to [3, p.343–344], we find that  $I'_1, I'_2$  are estimated as

$$\begin{aligned} I'_1 &\ll y^{1-\sigma}(\log y)^r \|\varphi^{(l+1)}\|_1 \times \\ &\quad \times \int_{L_{\pm 1}} |t|^{\frac{1}{2}-(\sigma+u)-\frac{k-1}{2}} (1+|t+v|)^{\sigma+u-\frac{1}{2}+\frac{k-1}{2}} \frac{(\log |v|)^{m-r}}{|v|^{l+1}} dv + \\ &\quad + y^{1-\sigma}(\log y)^r (\log |t|)^{m-r} \|\varphi^{(l+1)}\|_1 |t|^{-\frac{l+1}{2}} \int_{C_1} |t|^u \left( \frac{x}{2\pi y} \right)^u |dw| \\ &\ll y^{1-\sigma}(\log y)^r (\log |t|)^{m-r} |t|^{-\frac{l}{2}} \|\varphi^{(l+1)}\|_1, \end{aligned} \quad (3.15)$$

$$I'_2 \ll y^{1-\sigma}(\log y)^r (\log |t|)^{m-r} |t|^{-\frac{l}{2}} \|\varphi^{(l+1)}\|_1, \quad (3.16)$$

under the condition  $x/(2\pi y) = 1/|t|$ . From (3.8)–(3.10), (3.15) and (3.16), the proof of Proposition 3.2 is completed.  $\square$

We use (1.4) and combine the result Propositions 3.1 and 3.2. Let  $y_1, y_2$  be the positive numbers satisfying  $x/(2\pi y_2) = 1/|t|$ ,  $(1/x)/(2\pi y_2) = 1/|t|$  respectively. Using Remark 3.3, the main term of (1.7) is obtained. Then under the condition  $(2\pi)^2 y_1 y_2 = |t|^2$ , the proof of Theorem 1.1 is completed.

## 4 Proof of Theorem 1.2

To get the approximate functional equation for  $L_f^{(m)}(s)$  without characteristic functions, we introduce new functions  $\xi$ ,  $\psi_\alpha$  and  $\psi_{0\alpha}$ . Let  $\xi$  be the function defined by  $\xi(\rho) = 1$  when  $\rho \in [0, 1]$  and  $\xi(\rho) = 0$  when  $\rho \in [1, \infty)$ . For  $\alpha \in \mathbb{R}_{\geq 0}$  and  $\varphi \in \mathcal{R}$ , let  $\psi_\alpha$  be the function defined by

$$\psi_\alpha(\rho) = \begin{cases} 1, & \rho \in [0, 1 - 1/(2|t|^\alpha)], \\ \varphi(1 + (\rho - 1)|t|^\alpha), & \rho \in [1 - 1/(2|t|^\alpha), 1 + 1/|t|^\alpha], \\ 0, & \rho \in [1 + 1/|t|^\alpha, \infty), \end{cases}$$

and  $\psi_{0\alpha}$  is defined by  $\psi_{0\alpha}(\rho) = 1 - \psi_\alpha(1/\rho)$ .

**Remark 4.1.** From [3, (12)–(15)], we see that  $\psi_\alpha, \psi_{0\alpha} \in \mathcal{R}$ ,  $\xi \notin \mathcal{R}$ ,

$$(\psi_\alpha - \xi)(\rho) = 0, \quad (\psi_{0\alpha} - \xi)(\rho) = 0, \quad \psi_\alpha^{(j)}(\rho) = 0, \quad \psi_{0\alpha}^{(j)}(\rho) = 0.$$

for  $j \in \mathbb{Z}_{\geq 1}$  and  $\rho \in [0, 1 - 1/(2|t|^\alpha)] \cup [1 + 1/|t|^\alpha, \infty)$ , and

$$\psi_\alpha^{(j)}(\rho) \ll |t|^{\alpha j}, \quad \psi_{0\alpha}^{(j)}(\rho) \ll |t|^{\alpha j}, \quad \|\psi_\alpha^{(j)}\|_1 \ll |t|^{\alpha(j-1)}, \quad \|\psi_{0\alpha}^{(j)}\|_1 \ll |t|^{\alpha(j-1)}$$

for  $j \in \mathbb{Z}_{\geq 0}$  and  $\rho \in [0, \infty)$ .

Let  $M_\varphi(s)$  be the first sum on the right-hand side of (1.7). Setting  $y_1 = y_2 = |t|/(2\pi)$  and replacing  $\varphi \mapsto \psi_\alpha$  in Theorem 1.1, we can write

$$L_f^{(m)}(s) = M_\xi(s) + O(M_{\psi_\alpha - \xi}(s) + R_{\psi_\alpha}(s)). \quad (4.1)$$

Then we have

$$\begin{aligned} M_\xi(s) &= \sum_{n \leq \frac{|t|}{2\pi}} \frac{\lambda_f(n)(-\log n)^m}{n^s} + \\ &+ \sum_{r=0}^m (-1)^{m-r} \binom{m}{r} \chi_f^{(m-r)}(s) \sum_{n \leq \frac{|t|}{2\pi}} \frac{\lambda_f(n)(-\log n)^r}{n^{1-s}} \end{aligned} \quad (4.2)$$

and

$$\begin{aligned} &M_{\psi_\alpha - \xi}(s) + R_{\psi_\alpha}(s) \\ &\ll \sum_{\frac{|t|}{2\pi} \frac{1}{1+\frac{1}{|t|^\alpha}} \leq n \leq \frac{|t|}{2\pi} (1+\frac{1}{|t|^\alpha})} \frac{|\lambda_f(n)|(\log n)^m}{n^\sigma} |S_{\psi_\alpha}^{(0)}(s)| + \\ &+ \sum_{r=0}^m \sum_{\frac{|t|}{2\pi} \frac{1}{1+\frac{1}{|t|^\alpha}} \leq n \leq \frac{|t|}{2\pi} (1+\frac{1}{|t|^\alpha})} \frac{|\lambda_f(n)|(\log n)^r}{n^\sigma} |S_{\psi_{0\alpha}}^{(m-r)}(1-s)| + \\ &+ |t|^{1-\sigma+(\alpha-\frac{1}{2})l} (\log |t|)^m. \end{aligned} \quad (4.3)$$

where  $S_{\psi_\alpha}^{(r)}(s)$  is given by

$$S_{\psi_\alpha}^{(r)}(s) = (\psi_\alpha - \xi) \left( \frac{2\pi n}{|t|} \right) \frac{\chi_f^{(r)}}{\chi_f} (1-s) +$$

$$+ \sum_{j=1}^l \psi_{\alpha}^{(j)} \left( \frac{2\pi n}{|t|} \right) \left( -\frac{2\pi n}{|t|} \right)^j \gamma_j^{(r)} \left( s, \frac{1}{|t|} \right),$$

and we used Remarks 3.3, 4.1, (1.4) and the fact  $1 - 1/(2|t|^{\alpha}) \geq 1/(1 + 1/|t|^{\alpha})$  for  $\alpha \in \mathbb{R}_{\geq 0}$ . Using Remarks 3.3 and 4.1, in the case of  $n \in [|t|/(2\pi(1 + |t|^{-\alpha})) , (1 + |t|^{-\alpha})|t|/(2\pi)]$  the sum  $S_{\psi_{\alpha}}^{(r)}(s)$  is estimated as follows under the condition  $\alpha \leq 1/2$ :

$$S_{\psi_{\alpha}}^{(r)}(s) \ll (\log |t|)^r + \sum_{j=1}^l |t|^{(\alpha - \frac{1}{2})j} (\log |t|)^r \ll (\log |t|)^r \ll |t|^{\varepsilon}. \quad (4.4)$$

Deligne's estimate  $|\lambda_f(n)| \leq d(n) \ll n^{\varepsilon}$  (see [2]) gives

$$\sum_{\frac{|t|}{2\pi} \frac{1}{1 + \frac{1}{|t|^{\alpha}}} \leq n \leq \frac{|t|}{2\pi} (1 + \frac{1}{|t|^{\alpha}})} \frac{|\lambda_f(n)| (\log n)^r}{n^{\sigma}} \ll |t|^{1 - \sigma - \alpha + \varepsilon}. \quad (4.5)$$

Therefore combining (4.3)–(4.5), we obtain the following estimate:

$$M_{\psi_{\alpha} - \xi}(s) + R_{\psi_{\alpha}}(s) = O(|t|^{1 - \sigma - \alpha + \varepsilon}) + O(|t|^{1 - \sigma + (\alpha - \frac{1}{2})l + \varepsilon}) = O(|t|^{\frac{1}{2} - \sigma + \varepsilon}), \quad (4.6)$$

where we put  $\alpha = 1/2 - \varepsilon$  and take  $l \geq 1/(2\varepsilon)$ . Combining (4.1), (4.2) and (4.6), we obtain the assertion of Theorem 1.2.

## 5 Proof of Theorem 1.3

Putting  $y_1 = y_2 = |t|/(2\pi)$  in Theorem 1.1 and writing  $\varphi = \varphi_1 + \varphi_2$ ,  $\varphi_0 = \varphi_{01} + \varphi_{02}$  where  $\varphi_1, \varphi_2, \varphi_{01}, \varphi_{02}$  are defined by (2.6), we obtain the following formula:

$$\int_0^T |L_f^{(m)}(s)|^2 dt = \int_1^T \left| \sum_{r=1}^5 S_r(s) \right|^2 dt + O(1) = \sum_{1 \leq \mu, \nu \leq 5} I_{\mu, \nu} + O(1), \quad (5.1)$$

where  $S_r(s)$  are given by

$$S_1(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n) (-\log n)^m}{n^s} \varphi_1 \left( \frac{2\pi n}{t} \right),$$

$$\begin{aligned}
S_2(s) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^m}{n^{1-s}} \varphi_2\left(\frac{2\pi n}{t}\right), \\
S_3(s) &= \sum_{r=0}^m (-1)^r \binom{m}{r} \chi_f^{(m-r)}(s) \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^r}{n^{1-s}} \varphi_{01}\left(\frac{2\pi n}{t}\right), \\
S_4(s) &= \sum_{r=0}^m (-1)^r \binom{m}{r} \chi_f^{(m-r)}(s) \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^r}{n^{1-s}} \varphi_{02}\left(\frac{2\pi n}{t}\right), \\
S_5(s) &= R_{\varphi}(s),
\end{aligned}$$

and  $I_{\mu,\nu}$  ( $\mu, \nu \in \{1, \dots, 5\}$ ) are given by

$$I_{\mu,\nu} = \int_1^T S_{\mu}(s) \overline{S_{\nu}(s)} dt.$$

First we consider the integral  $I_{\mu,\nu}$  in the case of  $\mu = \nu$ . In the case of  $(\mu, \nu) = (1, 1)$ , applying (a), (c) of Lemma 2.5, we get

$$\begin{aligned}
I_{1,1} &= \sum_{n_1, n_2=1}^{\infty} \frac{\overline{\lambda_f(n_1)} \lambda_f(n_2) (\log n_1 \log n_2)^m}{(n_1 n_2)^{\sigma}} \times \\
&\quad \times \int_1^T \overline{\varphi_1\left(\frac{2\pi n_1}{t}\right)} \varphi_1\left(\frac{2\pi n_2}{t}\right) \left(\frac{n_1}{n_2}\right)^{it} dt \\
&= T \sum_{n \leq \frac{\delta_1}{2\pi} T} \frac{|\lambda_f(n)|^2 (\log n)^{2m}}{n^{2\sigma}} + O\left(\sum_{n \leq \frac{\delta_1}{2\pi} T} \frac{|\lambda_f(n)|^2 (\log n)^{2m}}{n^{2\sigma-1}}\right) + \\
&\quad + \frac{1}{i} \sum_{\substack{n_1, n_2 < \frac{\delta_1}{2\pi} T, \\ n_1 \neq n_2}} \frac{\overline{\lambda_f(n_1) \varphi_1(2\pi n_1/T) n_1^{-iT}} \lambda_f(n_2) \varphi_1(2\pi n_2/T) n_2^{-iT}}{(n_1 n_2)^{\sigma}} \times \\
&\quad \times \frac{\log(n_1/n_2)}{(\log n_1 \log n_2)^m} + O\left(\sum_{n_1 < n_2 \leq \frac{\delta_1}{2\pi} T} \frac{|\lambda_f(n_1) \lambda_f(n_2)| (\log n_1 \log n_2)^m}{(n_1 n_2)^{\sigma} n_2 (\log(n_1/n_2))^2}\right) \\
&=: U_1 + O(U_2) + U_3 + O(U_4). \tag{5.2}
\end{aligned}$$

Here we shall calculate the right-hand side of (5.2). Using partial summation



and (1.9), we obtain the approximate formula for  $U_1$  as

$$U_1 = \begin{cases} \frac{C_f}{2m+1} T(\log T)^{2m+1} + O(T), & \sigma = 1/2, \\ T \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2 (\log n)^{2m}}{n^{2\sigma}} + O(T^{2(1-\sigma)} (\log T)^{2m}), & \sigma \in (1/2, 1]. \end{cases} \quad (5.3)$$

The result (1.9), the estimates (d), (c) of Lemma 2.6 imply that

$$U_j = \begin{cases} O(T^{2(1-\sigma)} (\log T)^{2m}), & \sigma \in [1/2, 1), \\ O((\log T)^{2m+2}), & \sigma = 1 \end{cases} \quad (5.4)$$

for  $j = 2, 3, 4$  respectively. From (5.2)–(5.4), the error term and the main term of  $I_{1,1}$  correspond to those of the right-hand side of (1.10) when  $\sigma \in (1/2, 1]$ . However, the main term of the right-hand side of (1.10) is not obtained completely when  $\sigma = 1/2$ . In the case of  $(\mu, \nu) = (2, 2)$ , applying (b), (c) of Lemma 2.5 and (a), (c), (d) of Lemma 2.6, we obtain

$$\begin{aligned} I_{2,2} &= \frac{1}{i} \sum_{\substack{n_1, n_2 < \frac{T}{\pi}, \\ n_1 \neq n_2}} \frac{\overline{\lambda_f(n_1) \varphi_2(2\pi n_1/T) n_1^{-iT}} \lambda_f(n_2) \varphi_2(2\pi n_2/T) n_2^{-iT}}{(n_1 n_2)^\sigma} \times \\ &\quad \times \frac{(\log n_1 \log n_2)^m}{\log(n_1/n_2)} + O\left( \sum_{n_1 < n_2 \leq \frac{T}{\pi}} \frac{|\lambda_f(n_1) \lambda_f(n_2)| (\log n_1 \log n_2)^m}{(n_1 n_2)^\sigma n_2 (\log(n_1/n_2))^2} \right) + \\ &\quad + O\left( \sum_{n \leq \frac{T}{\pi}} \frac{|\lambda_f(n)|^2 (\log n)^{2m}}{n^{2\sigma-1}} \right) \\ &= \begin{cases} O(T^{2(1-\sigma)} (\log T)^{2m}), & \sigma \in [1/2, 1), \\ O((\log T)^{2m+1}), & \sigma = 1. \end{cases} \end{aligned} \quad (5.5)$$

Next we consider the case  $(\mu, \nu) = (3, 3)$ . Using (2.17) and the condition  $r_1 + r_2 = r$ , we obtain the following formula:

$$(\overline{\chi_f^{(m-r_1)}} \chi_f^{(m-r_2)})(s) = (2\pi)^{4\sigma-2} (-2)^{2m-r} \frac{(\log \frac{t}{2\pi})^{2m-r}}{t^{4\sigma-2}} + M(t).$$

where  $M(t)$  is given by  $M(t) = O((\log t)^{2m-r}/t^{4\sigma-1})$ . Then  $I_{3,3}$  is written as

$$\begin{aligned} I_{3,3} &= \sum_{r=0}^{2m} \sum_{r_1+r_2=r} (-1)^r \binom{m}{r_1} \binom{m}{r_2} \sum_{n_1, n_2=1}^{\infty} \frac{\overline{\lambda_f(n_1)} \lambda_f(n_2) (\log n_1)^{r_1} (\log n_2)^{r_2}}{(n_1 n_2)^{1-\sigma}} \times \\ &\quad \times \int_1^T \overline{\varphi_{01}\left(\frac{2\pi n_1}{t}\right)} \varphi_{01}\left(\frac{2\pi n_2}{t}\right) \left(\frac{n_1}{n_2}\right)^{it} (\chi_f^{(m-r_1)} \chi_f^{(m-r_2)})(s) dt \\ &= I_{3,3}^+ + I_{3,3}^-, \end{aligned} \tag{5.6}$$

where  $I_{3,3}^+$ ,  $I_{3,3}^-$  are defined by

$$\begin{aligned} I_{3,3}^+ &:= (2\pi)^{4\sigma-2} \sum_{r=0}^{2m} (-2)^{2m-r} \sum_{r_1+r_2=r} \binom{m}{r_1} \binom{m}{r_2} \times \\ &\quad \times \sum_{n_1, n_2=1}^{\infty} \frac{\overline{\lambda_f(n_1)} \lambda_f(n_2) (\log n_1)^{r_1} (\log n_2)^{r_2}}{(n_1 n_2)^{1-\sigma}} \times \\ &\quad \times \int_1^T \overline{\varphi_{01}\left(\frac{2\pi n_1}{t}\right)} \varphi_{01}\left(\frac{2\pi n_2}{t}\right) \left(\frac{n_1}{n_2}\right)^{it} \frac{(\log \frac{t}{2\pi})^{2m-r}}{t^{4\sigma-2}} dt, \\ I_{3,3}^- &:= \sum_{r=0}^{2m} \sum_{r_1+r_2=r} \binom{m}{r_1} \binom{m}{r_2} \sum_{n_1, n_2=1}^{\infty} \frac{\overline{\lambda_f(n_1)} \lambda_f(n_2)}{(n_1 n_2)^{1-\sigma}} \times \\ &\quad \times (\log n_1)^{r_1} (\log n_2)^{r_2} \int_1^T \overline{\varphi_{01}\left(\frac{2\pi n_1}{t}\right)} \varphi_{01}\left(\frac{2\pi n_2}{t}\right) \left(\frac{n_1}{n_2}\right)^{it} M(t) dt. \end{aligned}$$

respectively. Here we shall approximate  $I_{3,3}^+$  and  $I_{3,3}^-$ . In order to estimate  $I_{3,3}^-$ , we use the fact that  $(n_1 n_2)^{1-\sigma} n_2^{4\sigma-2} = (n_1 n_2)^{\sigma} (n_2/n_1)^{2\sigma-1} \gg (n_1 n_2)^{\sigma}$  for  $\sigma \in \mathbb{R}_{\geq 1/2}$  and  $n_1 \leq n_2$ . Then using (d) of Lemma 2.5 and (a), (b) of Lemma 2.6, we see that

$$\begin{aligned} I_{3,3}^- &\ll \sum_{r=0}^{2m} \sum_{r_1+r_2=r} \sum_{n_1 \leq n_2 \leq \frac{\delta_1}{2\pi} T} \frac{|\lambda_f(n_1) \lambda_f(n_2)| (\log n_1)^{r_1} (\log n_2)^{r_2}}{(n_1 n_2)^{1-\sigma}} \times \\ &\quad \times \begin{cases} |\log(T/n_2)| (\log T)^{2m-r}, & \sigma = 1/2, \\ (\log n_2)^{2m-r} / n_2^{4\sigma-2}, & \sigma \in (1/2, 1] \end{cases} \end{aligned}$$

$$\ll \begin{cases} T^{2(1-\sigma)}(\log T)^{2m}, & \sigma \in [1/2, 1), \\ (\log T)^{2m+2}, & \sigma = 1. \end{cases} \quad (5.7)$$

The formula (a), (c) of Lemma 2.5 imply that

$$\begin{aligned} I_{3,3}^+ &= \begin{cases} \frac{(2\pi)^{4\sigma-2}}{3-4\sigma} T^{3-4\sigma} \sum_{r=0}^{2m} (2 \log \frac{T}{2\pi})^{2m-r} \times \\ \times \sum_{r_1+r_2=r} \binom{m}{r_1} \binom{m}{r_2} \sum_{n \leq \frac{\delta_1}{2\pi} T} \frac{|\lambda_f(n)|^2 (\log n)^r}{n^{2(1-\sigma)}}, & \sigma \in [1/2, 4/3), \\ 0, & \sigma \in [3/4, 1], \end{cases} + \\ &+ O \left( \sum_{r=0}^{2m} \sum_{n \leq \frac{\delta_1}{2\pi} T} \frac{|\lambda_f(n)|^2 (\log n)^r}{n^{2(1-\sigma)}} \times \begin{cases} T^{3-4\sigma} (\log T)^{2m-r}, \\ \sigma \in [1/2, 3/4), \\ |\log(T/n)| (\log T)^{2m-r}, \\ \sigma = 3/4, \\ (\log T)^{2m-r} / n^{4\sigma-3}, \\ \sigma \in (3/4, 1]. \end{cases} \right) + \\ &+ O \left( \sum_{r=0}^{2m} \sum_{\frac{\delta}{2\pi} T < n \leq \frac{\delta_1}{2\pi} T} \frac{|\lambda_f(n)|^2 (\log n)^r (\log T)^{2m-r}}{n^{2(1-\sigma)} n^{4\sigma-3}} \right) + \\ &+ \frac{(2\pi)^{4-2\sigma}}{i} \sum_{r=0}^{2m} (2 \log \frac{T}{2\pi})^{2m-r} \sum_{r_1+r_2=r} \binom{m}{r_1} \binom{m}{r_2} \times \\ &\times \sum_{\substack{n_1, n_2 \leq \frac{\delta_1}{2\pi} T, \\ n_1 \neq n_2}} \frac{(\log n_1)^{r_1} (\log n_2)^{r_2}}{(n_1 n_2)^\sigma \log(n_1/n_2)} \times \\ &\times \frac{\lambda_f(n_1) \varphi_{01}(2\pi n_1/T) (n_1/T)^{2\sigma-1}}{\overline{n_1^{iT}}} \frac{\lambda_f(n_2) \varphi_{01}(2\pi n_2/T) (n_2/T)^{2\sigma-1}}{n_2^{iT}} + \\ &+ O \left( \sum_{r=0}^{2m} \sum_{r_1+r_2=r} \sum_{n_1 < n_2 \leq \frac{\delta_1}{2\pi} T} \frac{|\lambda_f(n_1) \lambda_f(n_2)| (\log n_1)^{r_1} (\log n_2)^{r_2}}{(n_1 n_2)^{1-\sigma}} \times \right. \\ &\times \left. \frac{(\log n_2)^{2m-r}}{n_2^{4\sigma-1} (\log(n_1/n_2))^2} \right) \\ &=: V_1 + O(V_2) + O(V_3) + V_4 + O(V_5), \end{aligned} \quad (5.8)$$

A similar discussion to  $U_3$  gives that  $V_1$  is approximated as

$$V_1 = \begin{cases} (A_{f,m} - C_f/(2m+1))T(\log T)^{2m+1} + O(T(\log T)^{2m}), & \sigma = 1/2, \\ O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma \in (1/2, 1]. \end{cases} \quad (5.9)$$

To estimate  $V_4$  and  $V_5$ , we use the fact that  $(n_1 n_2)^{1-\sigma} n_2^{4\sigma-1} = (n_1 n_2)^\sigma n_2 (n_2/n_1)^{2\sigma-1} \gg (n_1 n_2)^\sigma n_2$  for  $\sigma \in \mathbb{R}_{\geq 1/2}$  and  $n_1 \leq n_2$ . Then the estimates (d), (c) of Lemma 2.6 give that

$$V_j = \begin{cases} O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma \in [1/2, 1), \\ O((\log T)^{2m+2}), & \sigma = 1 \end{cases} \quad (5.10)$$

for  $j = 4, 5$  respectively. By the fact that  $n^{2(1-\sigma)} \gg n^{2(1-\sigma)} n^{4\sigma-3} = n^{2\sigma-1}$  for  $\sigma \in \mathbb{R}_{\leq 3/4}$ , the estimate (b) of Lemma 2.6 when  $\sigma = 3/4$  and the formula (1.9), the sum  $V_2$  and  $V_3$  are estimated as

$$V_j = \begin{cases} O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma \in [1/2, 1), \\ O((\log T)^{2m+1}), & \sigma = 1 \end{cases} \quad (5.11)$$

for  $j = 2, 3$ . Therefore, from (5.6)–(5.11) the approximate formula for  $I_{3,3}$  is obtained. In the case of  $(\mu, \nu) = (4, 4)$ , by a similar discussion to the case of  $(\mu, \nu) = (3, 3)$  the integral  $I_{4,4}$  is approximated as

$$\begin{aligned} I_{4,4} = & O \left( \sum_{r=0}^{2m} \sum_{r_1+r_2=r} \sum_{n \leq \frac{T}{\pi}} \frac{|\lambda_f(n)|^2 (\log n)^{2m}}{n^{2\sigma-1}} \right) + \\ & + \frac{(2\pi)^{4-2\sigma}}{i} \sum_{r=0}^{2m} \left( 2 \log \frac{T}{2\pi} \right)^{2m-r} \sum_{r_1+r_2=r} \binom{m}{r_1} \binom{m}{r_2} \times \\ & \times \sum_{\substack{n_1, n_2 \leq \frac{T}{\pi}, \\ n_1 \neq n_2}} \frac{(\log n_1)^{r_1} (\log n_2)^{r_2}}{(n_1 n_2)^\sigma \log(n_1/n_2)} \times \\ & \times \frac{\lambda_f(n_1) \varphi_{02}(2\pi n_1/T) (n_1/T)^{2\sigma-1}}{n_1^{iT}} \frac{\lambda_f(n_2) \varphi_{02}(2\pi n_2/T) (n_2/T)^{2\sigma-1}}{n_2^{iT}} + \end{aligned}$$

$$\begin{aligned}
& + O \left( \sum_{r=0}^{2m} \sum_{r_1+r_2=r} \sum_{n_1 < n_2 \leq \frac{T}{\pi}} \frac{|\lambda_f(n_1)\lambda_f(n_2)|(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1 n_2)^{1-\sigma}} \times \right. \\
& \quad \left. \times \frac{(\log n_2)^{2m-r}}{n_2^{4\sigma-1}(\log(n_1/n_2))^2} \right) + \\
& + O \left( \sum_{r=0}^{2m} \sum_{r_1+r_2=r} \sum_{n_1 \leq n_2 \leq \frac{T}{\pi}} \frac{|\lambda_f(n_1)\lambda_f(n_2)|^2(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1 n_2)^{1-\sigma}} \times \right. \\
& \quad \left. \times \begin{cases} |\log(T/n)|(\log T)^{2m-r}, & \sigma = 1/2, \\ (\log n_2)^{2m-r}/n_2^{4\sigma-2}, & \sigma \in (1/2, 1] \end{cases} \right) \\
& = \begin{cases} O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma \in [1/2, 1), \\ O((\log T)^{2m+2}), & \sigma = 1, \end{cases} \tag{5.12}
\end{aligned}$$

where (b)–(d) of Lemma 2.5, the formula (1.9) and (b)–(d) of Lemma 2.6 were used. Finally we consider the case  $(\mu, \nu) = (5, 5)$ . Remarks 3.3, 4.1 and the formula (1.4) imply that

$$\begin{aligned}
R_\varphi(s) & \ll \sum_{\frac{t}{4\pi} \leq n \leq \frac{t}{\pi}} \frac{|\lambda_f(n)|(\log n)^m}{n^\sigma} \left( \frac{1}{|t|} + \sum_{j=2}^l \frac{1}{|t|^{\frac{j}{2}}} \right) + |\chi_f(s)| \sum_{r=0}^m \sum_{\frac{t}{4\pi} \leq n \leq \frac{t}{\pi}} 1 \times \\
& \quad \times \frac{|\lambda_f(n)|(\log n)^r}{n^{1-\sigma}} \left( \frac{(\log |t|)^{m-r}}{|t|} + \sum_{j=2}^l \frac{(\log |t|)^{m-r}}{|t|^{\frac{j}{2}}} \right) + \frac{(\log |t|)^m}{|t|^{\sigma-1+\frac{l}{2}}} \\
& \ll \frac{(\log t)^m}{t^\sigma}. \tag{5.13}
\end{aligned}$$

Hence we get

$$I_{5,5} \ll \int_1^T \frac{(\log t)^{2m}}{t^{2\sigma}} dt \ll \begin{cases} (\log T)^{2m+1}, & \sigma = 1/2, \\ 1, & \sigma \in (1/2, 1]. \end{cases} \tag{5.14}$$

Lastly we consider  $I_{\mu,\nu}$  in the case of  $\mu \neq \nu$ . Since  $I_{1,1}$  contains the main term of the mean value formula for  $L_f^{(m)}(s)$ , and Cauchy's inequality implies that  $|I_{\mu,\nu}| \leq I_{\mu,\mu} I_{\nu,\nu}$  for  $\mu, \nu \in \{1, \dots, 5\}$ , it follows that it is enough to consider  $I_{\mu,\nu}$  in the case of  $(\mu, \nu) = (1, 2), (1, 3), (1, 4), (1, 5)$ . First in

the case of  $(\mu, \nu) = (1, 2)$ , using (b), (c) of Lemma 2.5, (c), (d) of Lemma 2.6 and the estimate (5.3), we obtain

$$\begin{aligned}
I_{1,2} &= \sum_{n_1, n_2=1}^{\infty} \frac{\overline{\lambda_f(n_1)} \lambda_f(n_2) (\log n_1)^m (\log n_2)^m}{(n_1 n_2)^\sigma} \times \\
&\quad \times \int_1^T \overline{\varphi_1\left(\frac{2\pi n_1}{t}\right)} \varphi_2\left(\frac{2\pi n_2}{t}\right) \left(\frac{n_1}{n_2}\right)^{it} dt \\
&= \frac{1}{i} \sum_{\substack{n_1, n_2 < \frac{T}{\pi}, \\ n_1 \neq n_2}} \frac{\overline{\lambda_f(n_1)} \varphi_1(2\pi n_1/T) n_1^{-iT} \lambda_f(n_2) \varphi_2(2\pi n_2/T) n_2^{-iT}}{(n_1 n_2)^\sigma} \times \\
&\quad \times \frac{(\log n_1 \log n_2)^m}{\log(n_1/n_2)} + O\left(\sum_{n_1 < n_2 \leq \frac{T}{\pi}} \frac{|\lambda_f(n_1) \lambda_f(n_2)| (\log n_1 \log n_2)^m}{(n_1 n_2)^\sigma n_2 (\log(n_1/n_2))^2}\right) + \\
&\quad + O\left(\sum_{n < \frac{T}{\pi}} \frac{|\lambda_f(n)|^2 (\log n)^{2m}}{n^{2\sigma-1}}\right) \\
&= \begin{cases} O(T^{2(1-\sigma)} (\log T)^{2m}), & \sigma \in [1/2, 1), \\ O((\log T)^{2m+2}), & \sigma = 1. \end{cases} \tag{5.15}
\end{aligned}$$

Next we consider the case  $(\mu, \nu) = (1, 3)$ . From (e) of Lemma 2.5 and (a), (b) of Lemma 2.6, the integral  $I_{1,3}$  is estimated as

$$\begin{aligned}
I_{1,3} &= \sum_{r=0}^m (-1)^m \binom{m}{r} \sum_{n_1, n_2=1}^{\infty} \frac{\overline{\lambda_f(n_1)} \lambda_f(n_2) (\log n_1)^m (\log n_2)^m}{n_1^\sigma n_2^{1-\sigma}} \times \\
&\quad \times \int_1^T \overline{\varphi_1\left(\frac{2\pi n_1}{t}\right)} \varphi_{01}\left(\frac{2\pi n_2}{t}\right) (n_1 n_2)^{it} \chi_f^{(m-r)}(s) dt \\
&= O\left(\sum_{r=0}^{2m} \sum_{r_1+r_2=r} \sum_{n_1 \leq n_2 \leq \frac{\delta_1}{2\pi} T} \frac{|\lambda_f(n_1) \lambda_f(n_2)| (\log n_1)^{r_1} (\log n_2)^{r_2}}{n_1^\sigma n_2^{1-\sigma}} \times \right. \\
&\quad \times \begin{cases} |\log(T/n_2)| (\log T)^{2m-r}, & \sigma = 1/2, \\ (\log n_2)^{2m-r} / n_2^{2\sigma-1}, & \sigma \in (1/2, 1] \end{cases} \Bigg)
\end{aligned}$$

$$= \begin{cases} O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma \in [1/2, 1), \\ O((\log T)^{2m+2}), & \sigma = 1. \end{cases} \quad (5.16)$$

In the case of  $(\mu, \nu) = (1, 4)$ , a similar discussion to the case of  $(\mu, \nu) = (1, 3)$  gives that

$$I_{1,4} = \begin{cases} O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma \in [1/2, 1), \\ O((\log T)^{2m+2}), & \sigma = 1. \end{cases} \quad (5.17)$$

Finally we consider the case  $(\mu, \nu) = (1, 5)$ . The formula (1.9) and Cauchy's inequality imply that  $\sum_{n \leq x} |\lambda_f(n)| = O(x)$ . Then using the estimate (5.13) and partial summation we get

$$\begin{aligned} I_{1,5} &\ll \int_1^T \frac{(\log t)^m}{t^\sigma} \sum_{n \leq \frac{\delta_1}{2\pi} t} \frac{|\lambda_f(n)|(\log n)^m}{n^\sigma} dt \\ &\ll \int_1^T \frac{(\log t)^m}{t^\sigma} \begin{cases} t^{1-\sigma}(\log t)^m, & \sigma \in [1/2, 1), \\ (\log t)^{m+1}, & \sigma = 1 \end{cases} dt \\ &\ll \begin{cases} T^{2(1-\sigma)}(\log T)^{2m}, & \sigma \in [1/2, 1), \\ (\log T)^{2m+2}, & \sigma = 1. \end{cases} \end{aligned} \quad (5.18)$$

Therefore combining (5.1)–(5.18), we complete the proof of Theorem 1.3.  $\square$

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